

# Corrigendum to “Universal factorization of $3n - j$ ( $j > 2$ ) symbols...” [J. Phys. A: Math. Gen. 37 (2004) 3259]

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Ten values of  $12-j$  symbols of the first kind published earlier are challenged by values calculated with an independent Python program. The program first implements a narrow class of square roots of rational numbers, utilizing Python’s unlimited representation of big integers. Wigner’s  $3jm$  symbols,  $6-j$ ,  $9-j$ ,  $12-j$  and  $15-j$  symbols are then calculated by their familiar representations as sums over products of these.

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## I. UPDATE ON $12-j$ SYMBOLS

As remarked earlier [2, 12], the table of  $12-j$  symbols of the first kind listed by Wei and Dalgarno [22, 23] needs correction. Modified reference values are proposed in Table I, generated with the Python package delivered in the ancillary files. Each parameter set of angular momenta is fed into an instance of the `Wigner12j` class, the instance is called, and its value and floating point approximation are printed. Appendix A provides one manual check of the simplest case.

Table II prints reference values for the  $12-j$  symbols of the *second* kind, computed for a randomly selected set of half-integer input vectors.

## II. PYTHON3 PROGRAM

### A. Auxiliary class of square roots

The programming language Python uses an internal representation of integers with unlimited precision, and offers a representation of rational numbers on that basis in the `Fraction` class.

The module `surd` reproduced in the auxiliary files defines a `surd.Surd` class which represents a product of such a rational number by a positive square root of another such number, which suffices to calculate the Wigner symbols “exactly” [4]. Floating point representations are calculated on demand calling the `float` or `to_decimal` member functions. Multiplication and division of two `Surd` are forwarded to the `Fraction` implementation.

The task of keeping the number under the square root square-free is delegated to functionality provided through the Python package `NZMATH` [10], see `README.txt` in the auxiliary files.

Heuristically, the summations always reduce to members of a single quadratic field; sums (or differences) of square roots of rational numbers are apparently not needed [17]. (For the  $9-j$  symbols, this representability is a result of Wu’s factorizations [24].) As a backup, the class `surd.SurdVec` defines an exact representation of values of this kind, stored as vectors of the type `surd.Surd`, and also bestowed with the basic arithmetic binary functions.

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TABLE I: Reference values of 12- $j$  symbols of the first kind.

$\left\{ \begin{matrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{matrix} \right\}$	exact representation	decimal approximation
$\left\{ \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} \right\}$	1/54	0.0185185185185
$\left\{ \begin{matrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{matrix} \right\}$	$-7/3000 \cdot (7/3)^{(1/2)}$	-0.00356422554052
$\left\{ \begin{matrix} 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{matrix} \right\}$	$1/600 \cdot (7/3)^{(1/2)}$	0.00254587538609
$\left\{ \begin{matrix} 3 & 3 & 5 & 4 \\ 2 & 4 & 5 & 3 \\ 4 & 2 & 4 & 5 \end{matrix} \right\}$	$-1025/3841992 \cdot (5/13)^{(1/2)}$	-0.000165455319732
$\left\{ \begin{matrix} 4 & 3 & 5 & 6 \\ 3 & 5 & 6 & 4 \\ 2 & 4 & 3 & 5 \end{matrix} \right\}$	$1/143143 \cdot (17/11)^{(1/2)}$	8.68476363508e-06
$\left\{ \begin{matrix} 5 & 5 & 4 & 3 \\ 6 & 4 & 6 & 4 \\ 2 & 4 & 7 & 5 \end{matrix} \right\}$	$-4457/4969107 \cdot (19/5005)^{(1/2)}$	-5.5263583797e-05
$\left\{ \begin{matrix} 6 & 4 & 7 & 4 \\ 2 & 5 & 7 & 6 \\ 4 & 5 & 6 & 3 \end{matrix} \right\}$	$40201/1032470010 \cdot (19/130)^{(1/2)}$	1.48855379744e-05
$\left\{ \begin{matrix} 7 & 8 & 9 & 10 \\ 8 & 6 & 4 & 6 \\ 7 & 9 & 7 & 5 \end{matrix} \right\}$	$-167145847027/25854768127188 \cdot (1/285285)^{(1/2)}$	-1.21036254096e-05
$\left\{ \begin{matrix} 10 & 7 & 8 & 6 \\ 9 & 10 & 6 & 8 \\ 7 & 9 & 10 & 7 \end{matrix} \right\}$	$438509/135635524675 \cdot (957/910)^{(1/2)}$	3.31543358353e-06
$\left\{ \begin{matrix} 20 & 15 & 9 & 10 \\ 14 & 18 & 15 & 15 \\ 9 & 8 & 10 & 12 \end{matrix} \right\}$	$-28068059458324/13772930246561475 \cdot (2/1431494295)^{(1/2)}$	-7.61739062208e-08

### B. Formulas Implemented

The following standard representations are implemented in the `wigner3j` module [3, 14, 25]. The class `Wigner3jm` calculates one value of

$$\begin{aligned}
\left( \begin{matrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{matrix} \right) &= \delta_{m_1+m_2,-m} (-1)^{j_1-j_2-m} \sqrt{\frac{(j+j_1-j_2)!(j-j_1+j_2)!(j_1+j_2-j)!}{(j_1+j_2+j+1)!}} \\
&\times \sqrt{\frac{(j+m)!(j-m)!(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!}{(-1)^k}} \\
&\times \sum_k \frac{(-1)^k}{k!(j_1+j_2-j-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j-j_2+m_1+k)!(j-j_1-m_2+k)!}. \tag{1}
\end{aligned}$$

TABLE II: Reference values of 12- $j$  symbols of the second kind.

$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix}$	exact representation	decimal approximation
$\begin{bmatrix} 6 & 4 & 7 & 4 \\ 6 & 7 & 4 & 4 \\ 2 & 5 & 7 & 1 \end{bmatrix}$	$-28/23595 \cdot (1/195)^{(1/2)}$	$-8.49807860695\text{e-}05$
$\begin{bmatrix} 6 & 4 & 7 & 4 \\ 6 & 7 & 4 & 5 \\ 2 & 5 & 7 & 1 \end{bmatrix}$	$-7/47190 \cdot (119/2145)^{(1/2)}$	$-3.49387927903\text{e-}05$
$\begin{bmatrix} 6 & 4 & 7 & 4 \\ 6 & 7 & 5 & 3 \\ 2 & 5 & 7 & 1 \end{bmatrix}$	$4/212355 \cdot (14/39)^{(1/2)}$	$1.12857185282\text{e-}05$
$\begin{bmatrix} 5.5 & 4.5 & 6.5 & 3.5 \\ 6 & 6 & 5 & 3 \\ 1.5 & 4.5 & 6.5 & 0.5 \end{bmatrix}$	$1/572572 \cdot (57)^{(1/2)}$	$1.3185825425\text{e-}05$
$\begin{bmatrix} 5.5 & 4.5 & 6.5 & 3.5 \\ 6 & 6 & 5 & 4 \\ 1.5 & 4.5 & 6.5 & 0.5 \end{bmatrix}$	$1/40898 \cdot (19/14)^{(1/2)}$	$2.84846384914\text{e-}05$
$\begin{bmatrix} 5.5 & 4.5 & 6.5 & 3.5 \\ 6 & 6 & 6 & 3 \\ 1.5 & 4.5 & 6.5 & 0.5 \end{bmatrix}$	$-3/572572 \cdot (19/91)^{(1/2)}$	$-2.39412737646\text{e-}06$
$\begin{bmatrix} 10.5 & 4.5 & 11.5 & 3.5 \\ 4.5 & 16 & 4.5 & 16 \\ 8 & 7 & 14.5 & 19.5 \end{bmatrix}$	$-73/878560 \cdot (14147/21487898)^{(1/2)}$	$-2.13199726694\text{e-}06$
$\begin{bmatrix} 10.5 & 4.5 & 11.5 & 3.5 \\ 15 & 12 & 15 & 11 \\ 12.5 & 12.5 & 11.5 & 12.5 \end{bmatrix}$	$-41530704637/111423970458157500 \cdot (629/19)^{(1/2)}$	$-2.14456489187\text{e-}06$
$\begin{bmatrix} 10.5 & 4.5 & 11.5 & 3.5 \\ 11.5 & 9 & 14.5 & 3 \\ 16 & 16 & 5.5 & 4.5 \end{bmatrix}$	$-15680464081/328436652258000 \cdot (7/221)^{(1/2)}$	$-8.49689356966\text{e-}06$

The class `Wigner6j` uses

$$\begin{aligned}
\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{matrix} \right\} &= \Delta(j_1, j_2, j_3) \Delta(j_1, l_2, l_3) \Delta(l_1, j_2, l_3) \Delta(l_1, l_2, j_3) \\
&\times \sum_z (-1)^z \frac{(z+1)!}{(z-j_1-j_2-j_3)!(z-j_1-l_2-l_3)!(z-l_1-j_2-l_3)!(z-l_1-l_2-j_3)!} \\
&\times \frac{1}{(j_1+j_2+l_1+l_2-z)!(j_1+j_3+l_1+l_3-z)!(j_2+j_3+l_2+l_3-z)!},
\end{aligned} \tag{2}$$

with triangular factors defined as

$$\Delta(j_1, j_2, j_3) \equiv \left[ \frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!} \right]^{1/2}. \tag{3}$$

The class `Wigner9j` implements

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{matrix} \right\} = \sum_x (2x+1)(-1)^{2x} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_3 & k_3 & x \end{matrix} \right\} \left\{ \begin{matrix} l_1 & l_2 & l_3 \\ j_2 & x & k_2 \end{matrix} \right\} \left\{ \begin{matrix} k_1 & k_2 & k_3 \\ x & j_1 & l_1 \end{matrix} \right\}. \tag{4}$$

TABLE III: Reference values of 15- $j$  symbols of the first kind.

	exact representation	decimal approximation
$\left\{ \begin{array}{cccccc} 3 & 3 & 2 & 4 & 2.5 & \\ & 4 & 4 & 5 & 2.5 & 5 \\ 2.5 & 2.5 & 4.5 & 4.5 & 3 & \end{array} \right\}$	$-16939/2904545952*(13/55)^{(1/2)}$	$-2.83530541752\text{e-}06$
$\left\{ \begin{array}{cccccc} 2 & 4.5 & 5 & 4.5 & 4 & \\ & 2.5 & 3.5 & 4.5 & 2.5 & 4.5 \\ 3.5 & 2 & 2.5 & 4 & 3.5 & \end{array} \right\}$	$-161153/4840909920*(1/429)^{(1/2)}$	$-1.60724745689\text{e-}06$
$\left\{ \begin{array}{cccccc} 4.5 & 3 & 3 & 5 & 4.5 & \\ & 3.5 & 1 & 5 & 1.5 & 4.5 \\ 2 & 2.5 & 2.5 & 2.5 & 1 & \end{array} \right\}$	$-13/1829520*(13/6)^{(1/2)}$	$-1.04592908944\text{e-}05$
$\left\{ \begin{array}{cccccc} 4.5 & 2.5 & 4.5 & 5 & 2.5 & \\ & 2 & 4 & 3.5 & 3.5 & 3 \\ 3.5 & 3.5 & 4.5 & 3 & 1.5 & \end{array} \right\}$	$13451/65197440*(1/231)^{(1/2)}$	$1.35743186372\text{e-}05$
$\left\{ \begin{array}{cccccc} 4 & 1 & 2.5 & 5 & 4.5 & \\ & 4 & 2.5 & 2.5 & 3.5 & 4.5 \\ 5 & 5 & 3.5 & 4 & 2.5 & \end{array} \right\}$	$323/13172544*(1/39)^{(1/2)}$	$3.92645442513\text{e-}06$
$\left\{ \begin{array}{cccccc} 0.5 & 3 & 3.5 & 5 & 4 & \\ & 2.5 & 4.5 & 3.5 & 3 & 3.5 \\ 2.5 & 2 & 4.5 & 5 & 3 & \end{array} \right\}$	$147899/512265600*(1/429)^{(1/2)}$	$1.39393139182\text{e-}05$

The results of this 9- $j$  calculation have been validated against other published values [8, 16, 19, 20]. Other values, without the factor  $(-)^{2x}$ , have also appeared [13]. In the class **Wigner12j**, the 12- $j$  symbols of the first kind are [1, 6, 7, 11]

$$\left\{ \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{array} \right\} = \sum_x (2x+1)(-1)^{R_n-x} \left\{ \begin{array}{ccc} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{array} \right\} \left\{ \begin{array}{ccc} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{array} \right\} \left\{ \begin{array}{ccc} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_4 & k_4 & x \\ j_1 & k_1 & l_4 \end{array} \right\}, \quad (5)$$

where  $R_n \equiv \sum_{i=1}^n (j_i + l_i + k_i)$  at  $n = 4$ .

The symmetric 12- $j$  symbols of the second kind are [15, 25]

$$\left[ \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{array} \right] = (-1)^{l_1-l_2-l_3+l_4} \sum_x (2x+1) \left\{ \begin{array}{ccc} k_1 & k_2 & x \\ j_3 & j_1 & l_1 \end{array} \right\} \left\{ \begin{array}{ccc} k_3 & k_4 & x \\ j_3 & j_1 & l_2 \end{array} \right\} \left\{ \begin{array}{ccc} k_1 & k_2 & x \\ j_4 & j_2 & l_3 \end{array} \right\} \left\{ \begin{array}{ccc} k_3 & k_4 & x \\ j_4 & j_2 & l_4 \end{array} \right\}. \quad (6)$$

Representative output of six evaluations with the program is gathered in Table II.

In the classes **Wigner15j** and **Wigner18j**, the symbols of the first and second kind are [25, (17.1),(17.2)]

$$\left\{ \begin{array}{cccccc} j_1 & j_2 & \dots & j_n \\ l_1 & l_2 & \dots & l_n \\ k_1 & k_2 & \dots & k_n \end{array} \right\} = \sum_x (2x+1)(-1)^{R_n+(n-1)x} \left\{ \begin{array}{ccc} j_n & k_n & x \\ j_1 & k_1 & l_n \end{array} \right\} \prod_{i=1}^{n-1} \left\{ \begin{array}{ccc} j_i & k_i & x \\ k_{i+1} & j_{i+1} & l_i \end{array} \right\} \quad (7)$$

and

$$\left[ \begin{array}{cccccc} j_1 & j_2 & \dots & j_n \\ l_1 & l_2 & \dots & l_n \\ k_1 & k_2 & \dots & k_n \end{array} \right] = \sum_x (2x+1)(-1)^{R_n+nx} \left\{ \begin{array}{ccc} j_n & k_n & x \\ k_1 & j_1 & l_n \end{array} \right\} \prod_{i=1}^{n-1} \left\{ \begin{array}{ccc} j_i & k_i & x \\ k_{i+1} & j_{i+1} & l_i \end{array} \right\} \quad (8)$$

at  $n = 5$  and  $n = 6$ , respectively. Tables III and IV show numerical results of these.

In the class **Wigner15j**, the symbols of the third kind are [25, (20.3)]

$$\left\{ \begin{array}{cccccc} k_1 & k'_1 & k & k' & k_2 & k'_2 \\ p_1 & & p & & p_2 & \\ j_1 & j'_1 & j & j' & j_2 & j'_2 \end{array} \right\} = \sum_x (2x+1)(-1)^{x+p-j-k'} \left\{ \begin{array}{ccc} k & j & x \\ j' & k' & p \end{array} \right\} \left\{ \begin{array}{ccc} k & j & x \\ k_1 & j_1 & p_1 \end{array} \right\} \left\{ \begin{array}{ccc} k' & j' & x \\ k'_1 & j'_1 & p_1 \end{array} \right\}, \quad (9)$$

TABLE IV: Reference values of 15- $j$  symbols of the second kind.

	exact representation	decimal approximation
$\left[ \begin{array}{cccccc} 3.5 & 3.5 & 5 & 0.5 & 2 & \\ & 6 & 6.5 & 5.5 & 2.5 & 5.5 \\ 5 & 4 & 5.5 & 6 & 6.5 & \end{array} \right]$	$346789/9717364800*(119/4290)^{(1/2)}$	5.94376514774e-06
$\left[ \begin{array}{cccccc} 4 & 1.5 & 2 & 3 & 3.5 & \\ & 3.5 & 2.5 & 4 & 2.5 & 0.5 \\ 4 & 1.5 & 1 & 3 & 3.5 & \end{array} \right]$	$-1157/3386880*(1/35)^{(1/2)}$	-5.7743024419e-05
$\left[ \begin{array}{cccccc} 2.5 & 1.5 & 1 & 2.5 & 3.5 & \\ & 2 & 0.5 & 1.5 & 2 & 4 \\ 5 & 4 & 4.5 & 5 & 5 & \end{array} \right]$	$47/118800*(1/231)^{(1/2)}$	2.6030075474e-05
$\left[ \begin{array}{cccccc} 4.5 & 2.5 & 4.5 & 5 & 2.5 & \\ & 2 & 4 & 3.5 & 3.5 & 3 \\ 3.5 & 3.5 & 4.5 & 3 & 1.5 & \end{array} \right]$	$10327/143434368*(1/42)^{(1/2)}$	1.11095459009e-05
$\left[ \begin{array}{cccccc} 4 & 1 & 2.5 & 5 & 4.5 & \\ & 4 & 2.5 & 2.5 & 3.5 & 4.5 \\ 5 & 5 & 3.5 & 4 & 2.5 & \end{array} \right]$	$8545/807288768*(1/7)^{(1/2)}$	4.00068296486e-06
$\left[ \begin{array}{cccccc} 0.5 & 3 & 3.5 & 5 & 4 & \\ & 2.5 & 4.5 & 3.5 & 3 & 3.5 \\ 2.5 & 2 & 4.5 & 5 & 3 & \end{array} \right]$	$-2909/8781696*(1/1365)^{(1/2)}$	-8.96600541253e-06
$\left[ \begin{array}{cccccc} 3.5 & 3 & 4.5 & 3.5 & 4.5 & \\ & 0.5 & 4.5 & 3 & 3 & 1 \\ 3.5 & 3 & 4.5 & 4.5 & 2.5 & \end{array} \right]$	$50741/717171840*(1/455)^{(1/2)}$	3.31688256263e-06

generating Table V.

The fourth kind is

$$\left\{ \begin{array}{cccccc} j_1 & k_1 & s_1 & k'_1 & j'_1 & \\ p & l & s & l' & p' & \\ j_2 & k_2 & s_2 & k'_2 & j'_2 & \end{array} \right\} = (-1)^{k_1+k_2-s_1-s_2+p+p'+2l'} \sum_x (2x+1) \times \left\{ \begin{array}{ccc} j_1 & j'_2 & x \\ l & s_2 & j_2 \\ s_1 & l' & j'_1 \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j'_2 & x \\ k'_2 & k_1 & p \end{array} \right\} \left\{ \begin{array}{ccc} j_2 & j'_1 & x \\ k'_1 & k_2 & p' \end{array} \right\} \left\{ \begin{array}{ccc} k_1 & k'_2 & x \\ k_2 & k'_1 & s \end{array} \right\}, \quad (10)$$

which leads to Table VI.

The fifth kind and Table VII are based on [25, (20.9)]

$$\left\{ \begin{array}{cccccc} k_1 & k'_1 & j_1 & l_1 & l'_1 & \\ k_2 & k'_2 & j_2 & l_2 & l'_2 & \\ k_3 & k'_3 & j_3 & l_3 & l'_3 & \end{array} \right\} = \sum_{x_1, x_2} (2x_1+1)(2x_2+1)(-1)^{j_1+j_2+k_1+k'_1-k_2+k'_2-l'_2+l_3+x_2} \times \left\{ \begin{array}{ccc} l_2 & l'_3 & x_1 \\ k'_2 & k'_3 & k_1 \end{array} \right\} \left\{ \begin{array}{ccc} l'_2 & l_3 & x_2 \\ k_2 & k_3 & k'_1 \end{array} \right\} \left\{ \begin{array}{ccc} l_2 & l'_3 & x_1 \\ l'_2 & l_3 & x_2 \\ j_2 & j_3 & j_1 \end{array} \right\} \left\{ \begin{array}{ccc} k'_2 & k'_3 & x_1 \\ k_3 & k_2 & x_2 \\ l_1 & l'_1 & j_1 \end{array} \right\}. \quad (11)$$

### III. SUMMARY

A list of 10 values of the 12- $j$  symbols of the first kind that appeared earlier in the literature has been corrected. Reference values for 12- $j$  symbols of both kinds and for 15- $j$  values of all five kinds have been computed with a

TABLE V: Reference values of 15- $j$  symbols of the third kind.

	exact representation	decimal approximation
$\left\{ \begin{matrix} 6.5 & 3.5 & 6 & & 3.5 & 1.5 & 3 \\ & 3 & & 4.5 & & 4.5 & \\ 3.5 & 4.5 & 6.5 & & 5 & 5 & 2.5 \end{matrix} \right\}$	$75443/19434729600*(17/33)^{(1/2)}$	$2.78617209378\text{e-}06$
$\left\{ \begin{matrix} 5.5 & & 6.5 & 5.5 & 5.5 & 3 & 4 \\ & 3.5 & & 6 & & 2 & \\ 5 & & 6 & 5 & & 3 & 1 & 6 \end{matrix} \right\}$	$-16529/2763375615*(34/77)^{(1/2)}$	$-3.97466705158\text{e-}06$
$\left\{ \begin{matrix} 2.5 & & 1.5 & 6.5 & 3.5 & 6 & 5 \\ & 5.5 & & 6 & & 6.5 & \\ 3 & & 6 & 3.5 & 6.5 & 3.5 & 2.5 \end{matrix} \right\}$	$27441/2821634816*(5/1547)^{(1/2)}$	$5.5289030385\text{e-}07$
$\left\{ \begin{matrix} 5 & 3 & 5 & & 2.5 & 2 & 2.5 \\ & 3 & & 4.5 & & 5 & \\ 4 & 1 & 2 & & 2.5 & 5 & 3.5 \end{matrix} \right\}$	$19267/8324316000*(1/2)^{(1/2)}$	$1.6366301271\text{e-}06$
$\left\{ \begin{matrix} 3.5 & & 3.5 & 5 & & 3.5 & 3.5 & 2 \\ & 4.5 & & 1.5 & & 2.5 & & \\ 2 & & 5 & 3 & & 1.5 & 3 & 4.5 \end{matrix} \right\}$	$211/1724800*(3/77)^{(1/2)}$	$2.41467661705\text{e-}05$
$\left\{ \begin{matrix} 4 & 4.5 & 4 & & 1.5 & 1 & 4 \\ & 4 & & 4.5 & & 4 & \\ 4 & 1.5 & 5 & & 3.5 & 4 & 4 \end{matrix} \right\}$	$-26003/146779776*(1/2310)^{(1/2)}$	$-3.68596532572\text{e-}06$

TABLE VI: Reference values of 15- $j$  symbols of the fourth kind.

	exact representation	decimal approximation
$\left\{ \begin{matrix} 6 & & 2 & 0.5 & 6 & 5 \\ 4 & 5.5 & & 5 & 5.5 & 3 \\ & 2.5 & 4.5 & 3 & 1.5 & 4.5 \end{matrix} \right\}$	$101/17249760*(7/143)^{(1/2)}$	$1.29544601101\text{e-}06$
$\left\{ \begin{matrix} 6.5 & & 5.5 & 4 & 6.5 & 6.5 \\ 6 & 3.5 & & 3 & 4.5 & 6 \\ & 5 & 2 & 6.5 & 2 & 4 \end{matrix} \right\}$	$-33049/4210858080*(7/429)^{(1/2)}$	$-1.0025547351\text{e-}06$
$\left\{ \begin{matrix} 6 & 5.5 & & 5 & 3.5 & 6 \\ & 4 & 5 & 6.5 & 4 & 6 \\ & 5.5 & 1.5 & 3 & 5.5 & 6.5 \end{matrix} \right\}$	$-110945249/1354301649805104*(95)^{(1/2)}$	$-7.98463566586\text{e-}07$
$\left\{ \begin{matrix} 2.5 & & 2.5 & 2 & 4 & 3.5 \\ 1 & 1.5 & & 2.5 & 3.5 & 3.5 \\ & 4 & 4.5 & 3.5 & 4 & 1 \end{matrix} \right\}$		0
$\left\{ \begin{matrix} 4 & & 1.5 & 1.5 & 2.5 & 2 \\ 2.5 & 4.5 & & 4 & 1.5 & 0.5 \\ & 1.5 & 1 & 3 & 4 & 3.5 \end{matrix} \right\}$	$-5/54432*(1/14)^{(1/2)}$	$-2.45500111986\text{e-}05$
$\left\{ \begin{matrix} 2 & 2 & 2.5 & 2 & 2 \\ 4 & 0.5 & 4 & 3.5 & 3 \\ & 5 & 3 & 4.5 & 4 & 2 \end{matrix} \right\}$	$139/370440*(1/165)^{(1/2)}$	$2.92115735985\text{e-}05$

program written in Python3, which is made available as ancillary material.

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TABLE VII: Reference values of 15- $j$  symbols of the fifth kind.

	exact representation	decimal approximation
$\left\{ \begin{matrix} 1 & 4.5 & 3.5 & 5 & 4.5 \\ 6.5 & 6 & 3.5 & 3 & 5.5 \\ 5 & 2 & 4 & 5 & 5 \end{matrix} \right\}$	$-1430315/13264202952*(1/462)^{(1/2)}$	$-5.01683187528\text{e-}06$
$\left\{ \begin{matrix} 0.5 & 2.5 & 3 & 5.5 & 4.5 \\ 2 & 2 & 6 & 2 & 5 \\ 6.5 & 2.5 & 3 & 3.5 & 1.5 \end{matrix} \right\}$	$17/288288*(1/2145)^{(1/2)}$	$1.27323540877\text{e-}06$
$\left\{ \begin{matrix} 4.5 & 5 & 5 & 4 & 6 \\ 3 & 4.5 & 6 & 6.5 & 0.5 \\ 5.5 & 4 & 4 & 4 & 1 \end{matrix} \right\}$	$-18433/218640708*(1/390)^{(1/2)}$	$-4.26906428456\text{e-}06$
$\left\{ \begin{matrix} 5 & 6.5 & 3.5 & 6 & 2.5 \\ 1 & 3.5 & 4.5 & 5.5 & 5 \\ 2.5 & 2.5 & 6 & 5.5 & 2.5 \end{matrix} \right\}$	$-19583/1205836632*(5/42)^{(1/2)}$	$-5.6033933163\text{e-}06$
$\left\{ \begin{matrix} 6 & 3 & 6 & 6 & 2 \\ 3 & 6.5 & 3.5 & 1 & 2.5 \\ 1.5 & 5 & 3.5 & 2 & 5.5 \end{matrix} \right\}$	$-103/4530240*(323/273)^{(1/2)}$	$-2.47306739633\text{e-}05$
$\left\{ \begin{matrix} 1.5 & 3 & 1.5 & 5 & 4.5 \\ 3.5 & 4 & 3.5 & 3.5 & 4 \\ 2 & 3 & 4 & 1.5 & 3.5 \end{matrix} \right\}$	$30607/1173553920*(1/5)^{(1/2)}$	$1.16636025701\text{e-}05$
$\left\{ \begin{matrix} 5 & 2.5 & 4.5 & 4 & 2.5 \\ 2 & 4 & 1 & 3.5 & 4.5 \\ 2 & 1.5 & 3.5 & 4.5 & 3 \end{matrix} \right\}$	$-27397/1746360000*(13/231)^{(1/2)}$	$-3.72164479622\text{e-}06$

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### Appendix A: Validation Example

The verification of the first entry in Table I may be done with pen and paper. Eq. (5) is

$$\left\{ \begin{array}{cccc} 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right\} = \sum_x (2x+1)(-1)^{12-x} \left\{ \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \begin{array}{c} x \\ 1 \end{array} \right\} \left\{ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \begin{array}{c} x \\ 1 \end{array} \right\} \left\{ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \begin{array}{c} x \\ 0 \end{array} \right\} \left\{ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \begin{array}{c} x \\ 1 \end{array} \right\}. \quad (\text{A1})$$

The factor  $\Delta(j_1, j_2, j_3)$  on the right hand side of (2) applied to the first 6- $j$  symbol on the right hand side implies that nonzero contributions may only emerge from  $x$  equal to  $-1$ ,  $0$  or  $1$ . The special value [18, 9.5.1]

$$\left\{ \begin{array}{ccc} 0 & b & c \\ d & e & f \end{array} \right\} = (-1)^{b+e+d} \frac{\delta_{bc}\delta_{ef}}{\sqrt{(2b+1)(2e+1)}} \quad (\text{A2})$$

sets that first factor to zero if  $x = -1$  or  $x = 0$ . Only  $x = 1$  might contribute:

$$\left\{ \begin{array}{cccc} 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right\} = 3 \times (-1)^{12-1} \left\{ \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right\}. \quad (\text{A3})$$

From (A2) by symmetry of the 6- $j$  symbol [18]

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & 0 \end{array} \right\} = (-1)^{a+b+c} \frac{\delta_{ae}\delta_{bd}}{\sqrt{(2a+1)(2b+1)}} \quad (\text{A4})$$

we conclude that three factors on the right hand side are

$$\left\{ \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right\} = \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right\} = \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right\} = -1/3. \quad (\text{A5})$$

Furthermore [18, Table 9.2]

$$\left\{ \begin{array}{ccc} a & b & c \\ 1 & c & b \end{array} \right\} = (-1)^{a+b+c+1} \frac{1}{2} \cdot \frac{-a(a+1) + b(b+1) + c(c+1)}{\sqrt{b(2b+1)(b+1)c(2c+1)(c+1)}} \quad (\text{A6})$$

leads to

$$\left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right\} = (-1)^4 \frac{1}{2} \times \frac{-2+2+2}{\sqrt{3 \times 2 \times 3 \times 2}} = 1/6. \quad (\text{A7})$$

Inserting this value and (A5) into (A3) generates the value  $1/54$  on the top of Table I.

### Appendix B: Book Errata

Corrigenda to the translation of the book by Yutsis *et al* [25] are:

- Our equation (10) corrects typographic errors (concerning primes on the right hand sides) in [25, (20.6)] and also in [5, (9)].
- All directions of the lines  $l_1, l_2, \dots, l_n$  for the symbols of the first kind in [25, Fig 17.1] should be reversed, as already reported [9].
- Immediately related to the previous bullet, the directions of the lines  $l_1, \dots, l_4$  for the 12- $j$  symbols of the first kind in [25, Fig. 19.1a] and [25, Fig. 19.1b] need to be reversed. (The five sign changes by the inversion of the five  $l$  for the 15- $j$  symbols of the first kind in [25, Fig. 20.2a-b], however, appear to be already compensated by the  $+-$  “handedness” in these figures.)